Numerical Algorithms and Simulations for Reflected Backward Stochastic Differential Equations with two Continuous Barriers

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Abstract

In this paper we study different algorithms for reflected backward stochastic differential equations (BSDE in short) with two continuous barriers based on binomial tree framework. We introduce numerical algorithms by penalization method and reflected method respectively. In the end simulation results are also presented.

Keywords: Backward Stochastic Differential Equations with two continuous barriers, Penalization method, Discrete Brownian motion, Numerical simulation

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1 Introduction

Non-linear backward stochastic differential equations (BSDEs in short) were firstly introduced by Pardoux and Peng ([21], 1990), who proved the existence and uniqueness of the adapted solution, under smooth square integrability assumptions on the coefficient and the terminal condition, plus that the coefficient $g(t,\omega,y,z)$ is (t,ω)-uniformly Lipschitz in (y,z). Then El Karoui, Kapoudjian, Pardoux, Peng and Quenez introduced the notion of reflected BSDE (RBSDE in short) ([11], 1997) with one continuous lower barrier. More precisely, a solution for such an equation associated to a coefficient g, a terminal value ξ, a continuous barrier L_t, is a triplet $(Y_t,Z_t,A_t)_{0\leq t\leq T}$ of adapted processes valued in R^{1+d+1}, which satisfies

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)ds+A_T-A_t+\int_t^T Z_s dB_s,\quad 0\leq t\leq T,\text{a.s.},$$

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and $Y_t \geq L_t$ a.s. for any $0 \leq t \leq T$. A_t is non-decreasing continuous, and B_t is a d-dimensional Brownian motion. The role of A_t is to push upward the process Y in a minimal way, in order to keep it above L. In this sense it satisfies $\int_0^T (Y_s - L_s)dA_s = 0$.

Following this paper, Cvitanic and Karatzas ([9], 1996) introduced the notion of reflected BSDE with two continuous barriers. In this case a solution of such an equation associated to a coefficient g, a terminal value ξ, a continuous lower barrier L_t and a continuous upper barrier U_t, with $L_t \leq U_t$ and $L_T \leq \xi \leq U_T$ a.s., is a quadruple $(Y_t, Z_t, A_t, K_t)_{0\leq t\leq T}$ of adapted processes, valued in R^{1+d+1}, which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dB_s, 0 \leq t \leq T, \text{a.s.},$$

and $L_t \leq Y_t \leq U_t$, a.s. for any $0 \leq t \leq T$. Here A_t and K_t are increasing continuous process, whose roles are to keep the process Y between L and U in such a way that

$$\int_0^T (Y_s - L_s)dA_s = 0 \text{ and } \int_0^T (Y_s - U_s)dK_s = 0.$$

In view to prove the existence and uniqueness of a solution, the method is based on a Picard-type iteration procedure, which requires at each step the solution of a Dynkin game problem. Furthermore, the authors proved the existence result by penalization method when the coefficient g does not depend on z. In 2004 ([16]), Lepeltier and San Martin relaxed in some sense the condition on the barriers, proved by a penalization method an existence result, without any assumption other than the square integrability one on L and U, but only when there exists a continuous semi-martingale with terminal value ξ, between L and U. More recently, Lepeltier and Xu ([18]) studied the case when the barriers are right continuous and left limit (RCLL in short), and proved the existence and uniqueness of solution in both Picard iteration and penalization method. In 2005, Peng and Xu considered the most general case when barriers are just $\mathbf{L}^2$-processes by penalization method, and studied a special penalization BSDE, which penalized with two barriers at the same time, and proved that the solutions of these equations converge to the solution of reflected BSDE.

The calculation and simulation of BSDEs is essentially different from those of SDEs (see [14]). When g is linear in y and z, we may solve the solution of BSDE by considering its dual equation, which is a forward SDE. However for nonlinear case of g, we can not find the solution explicitly. Here our numerical algorithms is based on approximate Brownian motion by random walk. This method is first considered by Peng and Xu [25]. The convergence of this type of numerical algorithms is proved by Briand, Delyon and Mémin in 2000 ([4]) and 2002 [5] . In 2002, Mémin, Peng and Xu studied the algorithms for reflected BSDE with one barrier and proved its convergence (cf. [20]). Recently Chassagneux also studied discrete-time approximation of doubly reflected BSDE in [6].

In this paper, we consider different numerical algorithms for reflected BSDE with two continuous barriers. The basic idea is to approximate a Brownian motion by random walks based on binary tree model. Compare with the one barrier case (cf. [20]), the additive barrier brings more difficulties in proving the convergence of algorithm, which requires us to get finer

estimation. When the Brownian motion is 1-dimensional, our algorithms have advantages in computer programming. In fact we developed a software package based on these algorithms for BSDE with two barriers. Furthermore it also contains programs for classical BSDEs and reflected BSDEs with one barrier. One significant advantage of this package is that the users have a very convenient user-machine interface. Any user who knows the basics of BSDE can run this package without difficulty.

This paper is organized as follows. In Section 2, we recall some classical results of reflected BSDE with two continuous barriers, and discretization for reflected BSDE. In Section 3, we introduce implicit and implicit-explicit penalization schemes and prove their convergence. In Section 4, we study implicit and explicit reflected schemes, and get their convergence. In Section 5, we present some simulations for reflected BSDE with two barriers. The proof of convergence of penalization solution is in Appendix.

We should point out that recently there have been many different algorithms for computing solutions of BSDEs and the related results in numerical analysis, for example [3], [4], [7], [8], [10], [13], [19], [26]. In contrast to these results, our methods can easily be realized by computer in 1-dimensional case. In the multi-dimensional case, the algorithms are still suitable, however to realize them by computer is difficult, since it will require larger amount of calculation than 1-dimensional case.

2 Preliminaries: Reflected BSDEs with two barriers and Basic discretization

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(B_t)_{t\geq 0}$ a 1-dimensional Brownian motion defined on a fixed interval $[0,T]$, with a fixed $T>0$. We denote by $\{\mathcal{F}_t\}_{0\leq t\leq T}$ the natural filtration generated by the Brownian motion B, i.e., $\mathcal{F}_t = \sigma\{B_s; 0\leq s\leq t\}$ augmented with all P-null sets of $\mathcal{F}$. Here we mainly consider 1-dimensional case, since the solution of reflected BSDE is 1-dimensional. In fact, we can also generalize algorithms in this paper to multi-dimensional Brownian motion, which will require a huge amount of calculation. We introduce the following spaces for $p\in[1,\infty)$:

- $\mathbf{L}^p(\mathcal{F}_t)$:={$\mathbb{R}$-valued $\mathcal{F}_t$–measurable random variables X s. t. $E[|X|^p]<\infty$};
- $\mathbf{L}^p_{\mathcal{F}}(0,t)$:={$\mathbb{R}$–valued and $\mathcal{F}_t$–adapted processes φ defined on $[0,t]$, s. t. $E\int_0^t|\varphi_s|^p ds<\infty$};
- $\mathbf{S}^p(0,t)$:={$\mathbb{R}$–valued and $\mathcal{F}_t$–adapted continuous processes φ defined on $[0,t]$, s. t. $E[\sup_{0\leq t\leq T}|\varphi_t|^2]<\infty$};
- $\mathbf{A}^p(0,t)$:={increasing processes in $\mathbf{S}^p(0,t)$ with $A(0)=0$}.

We are especially interested in the case $p=2$.

2.1 Reflected BSDE: Definition and convergence results

The random variable ξ is considered as terminal value, satisfying $\xi \in \mathbf{L}^2(\mathcal{F}_T)$. Let $g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a t-uniformly Lipschitz function in (y,z), i.e., there exists a fixed $\mu > 0$ such that

$$\begin{aligned} |g(t,y_1,z_1) - g(t,y_2,z_2)| &\leq \mu(|y_1 - y_2| + |z_1 - z_2|) \\ \forall t &\in [0,T], \forall (y_1,z_1),(y_2,z_2) \in \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{1}$$

And $g(\cdot,0,0)$ is square integrable.

The solution of our BSDE with two barriers is reflected between a lower barrier L and an upper barrier U, which are supposed to satisfy

Assumption 2.1 *L and U are $\mathcal{F}_t$-progressively measurable continuous processes valued in $\mathbb{R}$, such that*

$$E[\sup_{0\leq t\leq T}((L_t)^+)^2 + \sup_{0\leq t\leq T}((U_t)^-)^2] < \infty. \tag{2}$$

and there exists a continuous process $X_t = X_0 - \int_0^t \sigma_s dB_s + V_t^+ - V_t^-$ where $\sigma \in \mathbf{L}^2_{\mathcal{F}}(0,T)$, V^+ and V^- are $(\mathcal{F}_t)$-adapted continuous increasing processes with $E[|V_T^+|^2] + E[|V_T^-|^2] < \infty$ such that

$$L_t \leq X_t \leq U_t, \quad P\text{-a.s. for } 0 \leq t \leq T.$$

Remark 2.1 *Condition (2) permits us to treat situations when $U_t \equiv +\infty$ or $L_t \equiv -\infty$, $t \in [0,T]$, in such cases the corresponding reflected BSDE with two barriers becomes a reflected BSDE with a single lower barrier L or a single upper barrier U, respectively.*

Definition 2.1 *The solution of a reflected BSDE with two continuous barriers is a quadruple $(Y,Z,A,K) \in \mathbf{S}^2(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T) \times \mathbf{A}^2(0,T) \times \mathbf{A}^2(0,T)$ defined on $[0,T]$ satisfying the following equations*

$$\begin{aligned} -dY_t &= g(t,Y_t,Z_t)dt + dA_t - dK_t - Z_t dB_t,\ Y_T = \xi \\ L_t &\leq Y_t \leq U_t, \quad dA_t \geq 0, dK_t \geq 0, \quad dA_t \cdot dK_t = 0. \end{aligned} \tag{3}$$

and the reflecting conditions

$$\int_0^T (Y_t - L_t)dA_t = \int_0^T (Y_t - U_t)dK_t = 0. \tag{4}$$

To prove the existence of the solution, penalization method is important. Thanks to the convergence results of penalization solution in [16], [15] for continuous barriers' case and methods in [23], we have the following results, especially it gives the convergence speed of penalization solutions.

Theorem 2.1 *(a) There exists a unique solution (Y,Z,A,K) of reflected BSDE, i.e. it satisfies (3), (4). Moreover it is the limit of penalization solutions $(\widehat{Y}_t^{m,p},\widehat{Z}_t^{m,p},\widehat{A}_t^{m,p},\widehat{K}_t^{m,p})$ as $m\to\infty$ then $p\to\infty$, or equivalent as $q\to\infty$ then $m\to\infty$. Here the penalization solution $(\widehat{Y}_t^{m,p},\widehat{Z}_t^{m,p},\widehat{A}_t^{m,p},\widehat{K}_t^{m,p})$ with respect to two barriers L and U is defined, for $m\in\mathbb{N}$, $p\in\mathbb{N}$, as the solution of a classical BSDE*

$$\begin{aligned} -d\widehat{Y}_t^{m,p} &= g(t,\widehat{Y}_t^{m,p},\widehat{Z}_t^{m,p})dt+m(\widehat{Y}_t^{m,p}-L_t)^-dt-q(\widehat{Y}_t^{m,p}-U_t)^+dt-\widehat{Z}_t^{m,p}dB_t, \\ \widehat{Y}_T^{m,p} &= \xi. \end{aligned} \tag{5}$$

And we set $\widehat{A}_t^{m,p}=m\int_0^t(\widehat{Y}_s^{m,p}-L_s)^-ds$, $\widehat{K}_t^{m,p}=p\int_0^t(\widehat{Y}_s^{m,p}-U_s)^+ds$.

(b) Consider a special penalized BSDE for the reflected BSDE with two barriers: for any $p\in\mathbb{N}$,

$$\begin{aligned} -dY_t^p &= g(t,Y_t^p,Z_t^p)dt+p(Y_t^p-L_t)^-dt-p(Y_t^p-U_t)^+dt-Z_t^pdB_t, \\ Y_T^p &= \xi, \end{aligned} \tag{6}$$

with $A_t^p=\int_0^t p(Y_s^p-L_s)^-ds$ and $K_t^p=\int_0^t p(Y_s^p-U_s)^+ds$. Then we have, as $p\to\infty$, $Y_t^p\to Y_t$ in $\mathbf{S}^2(0,T)$, $Z_t^p\to Z_t$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$ and $A_t^p\to A_t$ weakly in $\mathbf{S}^2(0,T)$ as well as $K_t^p\to K_t$. Moreover there exists a constant C depending on ξ, $g(t,0,0)$, μ, L and U, such that

$$E[\sup_{0\le t\le T}|Y_t^p-Y_t|^2+\int_0^T(|Z_t^p-Z_t|^2)dt+\sup_{0\le t\le T}[(A_t-K_t)-(A_t^p-K_t^p)]^2]\le\frac{C}{\sqrt{p}}. \tag{7}$$

The proof is based on the results in [16] and [23], we put it in Appendix.

Remark 2.2 *In the following, we focus on the penalized BSDE as (7), which consider the penalization with respect to the two barriers at the same time. And p in superscribe always stands for the penalization parameter.*

Now we consider a special case: Assume that

Assumption 2.2 *L and U are Itô processes of the following form*

$$\begin{aligned} L_t &= L_0+\int_0^t l_sds+\int_0^t\sigma_s^ldB_s, \\ U_t &= U_0+\int_0^t u_sds+\int_0^t\sigma_s^udB_s. \end{aligned} \tag{8}$$

Suppose that l_s and u_s are right continuous with left limits (RCLL in short) processes, σ_s^l and σ_s^u are predictable with $E[\int_0^T[|l_s|^2+\left|\sigma_s^l\right|^2+|u_s|^2+|\sigma_s^u|^2]ds<\infty$.

It is easy to check that if $L_t\le U_t$, then Assumption 2.1 is satisfied. We may just set $X=L$ or U, with $\sigma_s=\sigma_s^l$ or σ_s^u and $V^\pm=\int_0 l_s^\pm ds$ or $\int_0 u_s^\pm ds$. Here $l_s^\pm$ (resp. $u_s^\pm$) is the positive or the negative part of l (resp. u). As Proposition 4.2 in [11], we have following proposition for two increasing processes, which can give us the integrability of the increasing processes by barriers.

Proposition 2.1 *Let (Y,Z,A,K) be a solution of reflected BSDE (3). Then $Z_t=\sigma_t^l$, a.s.-$dP\times dt$ on the set $\{Y_t=L_t\}$, $Z_t=\sigma_t^u$, a.s.-$dP\times dt$ on the set $\{Y_t=U_t\}$. And*

$$
\begin{aligned}
0 &\leq dA_t \leq 1_{\{Y_t=L_t\}}[g(t,L_t,\sigma_t^l)+l_t]^- dt,\\
0 &\leq dK_t \leq 1_{\{Y_t=U_t\}}[g(t,U_t,\sigma_t^u)+u_t]^+ dt.
\end{aligned}
$$

So there exist positive processes α and β, with $0\leq \alpha_t,\beta_t\leq 1$, such that $dA_t=\alpha_t 1_{\{Y_t=L_t\}}[g(t,L_t,\sigma_t^l)+l_t]^- dt$, $dK_t=\beta_t 1_{\{Y_t=U_t\}}[g(t,U_t,\sigma_t^u)+u_t]^+ dt$.

Proof. We can prove these results easily by using similar techniques as in Proposition 4.2 in [11], in view that on the set $\{L_t=U_t\}$, we have $\sigma_t^l=\sigma_t^u$ and $l_t=u_t$. So we omit the details of the proof here. $\square$

In the following, we will work under Assumption 2.2.

2.2 Approximation of Brownian motion and barriers

We use random walk to approximate the Brownian motion. Consider for each $j=1,2,\cdots$,

$$
B_t^n := \sqrt{\delta}\sum_{j=1}^{[t/\delta]}\varepsilon_j^n, \quad \text{for all } \ 0\leq t\leq T,\ \delta=\frac{T}{n},
$$

where $\{\varepsilon_j^n\}_{j=1}^n$ is a $\{1,-1\}$-valued i.i.d. sequence with $P(\varepsilon_j^n=1)=P(\varepsilon_j^n=-1)=0.5$, i.e., it is a Bernoulli sequence. We set the discrete filtration $\mathcal{G}_j^n:=\sigma\{\varepsilon_1^n,\cdots,\varepsilon_j^n\}$ and $t_j=j\delta$, for $0\leq j\leq n$. We denote by $\mathbf{D}_t$ the space of RCLL functions from $[0,t]$ in $\mathbb{R}$, endowed with the topology of uniform convergence, and we assume that:

Assumption 2.3 $\Gamma:\mathbf{D}_T\rightarrow\mathbf{R}$ *is K-Lipschitz. We consider $\xi=\Gamma(B)$, which is $\mathcal{F}_T$-measurable and $\xi^n=\Gamma(B^n)$, which is $\mathcal{G}_n^n$-measurable, such that*

$$
E[|\xi|^2]+\sup_n E[|\xi^n|^2]<\infty
$$

Now we consider the approximation of the barriers L and U. Notice that L and U are progressively measurable with respect to the filtration $(\mathcal{F}_t)$, which is generated by Brownian motion. So they can be presented as a functional of Brownian motion, i.e. for each $t\in[0,T]$, $L_t=\Psi_1(t,(B_s)_{0\leq s\leq t})$ and $U_t=\Psi_2(t,(B_s)_{0\leq s\leq t})$, where $\Psi_1(t,\cdot)$ and $\Psi_2(t,\cdot):\mathbf{D}_t\rightarrow\mathbf{R}$. And we assume that $\Psi_1(t,\cdot)$ and $\Psi_2(t,\cdot)$ are Lipschitz. Then we get the discretizaton of the barriers $L_j^n=\Psi_1(t_j,(B_s^n)_{0\leq s\leq t})$ and $U_t^n=\Psi_2(t_j,(B_s^n)_{0\leq s\leq t})$. If $L_t\leq U_t$, then $L_j\leq U_j$. On the other hand, we mainly consider barriers which are Itô processes and satisfy Assumption 2.2. So we have a natural approximation: for $1\leq j\leq n$,

$$
\begin{aligned}
L_j^n &= L_0+\delta\sum_{i=0}^{j-1}l_i+\sum_{i=0}^{j-1}\sigma_i^l\varepsilon_{i+1}^n\sqrt{\delta},\\
U_j^n &= U_0+\delta\sum_{i=0}^{j-1}u_i+\sum_{i=0}^{j-1}\sigma_i^u\varepsilon_{i+1}^n\sqrt{\delta}
\end{aligned}
$$

where $l_i = l_{t_i}$, $\sigma_i^l = \sigma_{t_i}^l$, $u_i = u_{t_i}$, $\sigma_i^u = \sigma_{t_i}^u$. Then L_j^n and U_j^n are discrete versions of L and U, with $\sup_n E[\sup_j ((L_j^n)^+)^2 + \sup_j ((U_j^n)^-)^2] < \infty$ and $L_j^n \le U_j^n$ still hold. In the following, we may use both approximations.

In this paper, we study two different types of numerical schemes. The first one is based on the penalization approach, whereas the second is to obtain the solution Y by reflecting it between L and U and get two reflecting processes A and K directly. Throughout this paper, n always stands for the discretization of the time interval. And process $(\phi_j^n)_{0\le j\le n}$ is a discrete process with $n+1$ values, for $\phi = L$, U, y^p, y, etc.

3 Algorithms based on Penalization BSDE and their Convergence

3.1 Discretization of Penalization BSDE and Penalization schemes

First we consider the discretization of penalized BSDE with respect to two discrete barriers L^n and U^n. After the discretization of time interval, we get the following discrete backward equation on the same interval $[t_j, t_{j+1}]$, for $0 \le j \le n-1$,

$$\begin{aligned} y_j^{p,n} &= y_{j+1}^{p,n} + g(t_j, y_j^{p,n}, z_j^{p,n})\delta + a_j^{p,n} - k_j^{p,n} - z_j^{p,n}\sqrt{\delta}\varepsilon_{j+1}^n, \\ a_j^{p,n} &= p\delta(y_j^{p,n} - L_j^n)^-, \quad k_j^{p,n} = p\delta(y_j^{p,n} - U_j^n)^+. \end{aligned} \tag{9}$$

The terminal condition is $y_n^{p,n} = \xi^n$. Since for a large fixed $p > 0$, (6) is in fact a classical BSDE. By numerical algorithms for BSDEs (cf. [24]), explicit scheme gives $z_j^{p,n} = \frac{1}{2\sqrt{\delta}}(y_{j+1}^n|_{\varepsilon_j=1} - y_{j+1}^n|_{\varepsilon_j=-1})$, and $y_j^{p,n}$ is solved from the inversion of the following mapping

$$\begin{aligned} &y_j^{p,n} = (\Theta^p)^{-1}(E[y_{j+1}^{p,n}|\mathcal{G}_j^n]), \\ &\text{where } \Theta^p(y) = y - g(t_j, y, z_j^{p,n})\delta - p\delta(y - L_j^n)^- + p\delta(y - U_j^n)^+, \end{aligned}$$

by substituting $E[y_{j+1}^{p,n}|\mathcal{G}_j^n] = \frac{1}{2}(y_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=1} + y_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=-1})$ into it. And increasing processes $a_j^{p,n}$ and $k_j^{p,n}$ will be obtained from (9).

In many cases, the inversion of the operator Θ^p is not easy to solve. So we apply the implicit–explicit penalization scheme to (9), replacing $y_j^{p,n}$ in g by $E[y_{j+1}^{p,n}|\mathcal{G}_j^n]$, and get

$$\begin{aligned} \bar{y}_j^{p,n} &= \bar{y}_{j+1}^{p,n} + g(t_j, E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \bar{z}_j^{p,n})\delta + \overline{a}_j^{p,n} - \overline{k}_j^{p,n} - \bar{z}_j^{p,n}\sqrt{\delta}\varepsilon_{j+1}^n \\ \overline{a}_j^{p,n} &= p\delta(\bar{y}_j^{p,n} - L_j^n)^-, \quad \overline{k}_j^{p,n} = p\delta(\bar{y}_j^{p,n} - U_j^n)^+. \end{aligned}$$

In the same way, we get $\bar{z}_j^{p,n} = \frac{1}{2\sqrt{\delta}}(\bar{y}_{j+1}^n|_{\varepsilon_j^n=1} - \bar{y}_{j+1}^n|_{\varepsilon_j^n=-1})$ and

$$\bar{y}_j^{p,n} = E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \bar{z}_j^{p,n})\delta + \overline{a}_j^{p,n} - \overline{k}_j^{p,n}. \tag{10}$$

Solving this equation, we obtain

$$\overline{y}_j^{p,n} = E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n] + g(t_j, E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n], \overline{z}_j^{p,n})\delta$$

$$
\begin{aligned}
&+\frac{p\delta}{1+p\delta}(E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g(t_j,E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\overline{z}_j^{p,n})\delta-L_j^n)^-\\
&-\frac{p\delta}{1+p\delta}(E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g(t_j,E[\overline{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\overline{z}_j^{p,n})\delta-U_j^n)^+.
\end{aligned}
$$

with $E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]=\frac{1}{2}(\bar{y}_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=1}+\bar{y}_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=-1})$. For increasing processes, we can get them from

$$
\begin{aligned}
\overline{a}_j^{p,n} &= \frac{p\delta}{1+p\delta}(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g(t_j,E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta-L_j^n)^-,\\
\overline{k}_j^{p,n} &= \frac{p\delta}{1+p\delta}(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g(t_j,E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta-U_j^n)^+.
\end{aligned}
$$

3.2 Convergence of penalization schemes and estimations

First we give the following lemma, which is proved in [20]. This Gronwall type lemma is classical but here it is given with more detailed formulation.

Lemma 3.1 *Let a, b and α be positive constants, $\delta b<1$ and a sequence $(v_j)_{j=1,\ldots n}$ of positive numbers such that, for every j*

$$
v_j+\alpha\le a+b\delta\sum_{i=1}^{j}v_i. \tag{11}
$$

Then

$$
\sup_{j\le n}v_j+\alpha\le a\mathcal{E}_\delta(b),
$$

where $\mathcal{E}_\delta(b)=1+\sum_{p=1}^{\infty}\frac{b^p}{p}(1+\delta)\cdots(1+(p-1)\delta)$, which is a convergent series.

Notice the $\mathcal{E}_\delta(b)$ is increasing in δ and $\delta<\frac{1}{b}$, so we can replace the right hand side of (11) by a constant depending on b.

We define the discrete solutions, $(Y_t^{p,n},Z_t^{p,n},A_t^{p,n},\ K_t^{p,n})$ by the implicit penalization scheme

$$
Y_t^{p,n}=y_{[t/\delta]}^{p,n},\ Z_t^{p,n}=z_{[t/\delta]}^{p,n},\ A_t^{p,n}=\sum_{m=0}^{[t/\delta]}a_m^{p,n},\ K_t^{p,n}=\sum_{m=0}^{[t/\delta]}k_m^{p,n},
$$

or $(\bar{Y}_t^{p,n},\bar{Z}_t^{p,n},\overline{A}_t^{p,n},\bar{K}_t^{p,n})$ by the implicit–explicit penalization scheme,

$$
\bar{Y}_t^{p,n}=\bar{y}_{[t/\delta]}^{p,n},\ \bar{Z}_t^{p,n}=\bar{z}_{[t/\delta]}^{p,n},\ \overline{A}_t^{p,n}=\sum_{m=0}^{[t/\delta]}\overline{a}_m^{p,n},\ \bar{K}_t^{p,n}=\sum_{m=0}^{[t/\delta]}\overline{k}_m^{p,n}.
$$

Let us notice that the laws of the solutions (Y^p,Z^p,A^p,K^p) and $(Y^{p,n},Z^{p,n},A^{p,n},K^{p,n})$ or $(\bar{Y}^{p,n},\bar{Z}^{p,n},\overline{A}^{p,n},\bar{K}^{p,n})$ to penalized BSDE depend only on $(\mathbf{P}_B,\Gamma^{-1}(\mathbf{P}_B),g,\Psi_1^{-1}(\mathbf{P}_B),\Psi_2^{-1}(\mathbf{P}_B))$ and $(\mathbf{P}_{B^n},\Gamma^{-1}(\mathbf{P}_{B^n}),g,\Psi_1^{-1}(\mathbf{P}_{B^n}),\Psi_2^{-1}(\mathbf{P}_{B^n}))$ where $\mathbf{P}_B$(resp. $\mathbf{P}_{B^n}$) is the probability introduced by B(resp. B^n), and $f^{-1}(\mathbf{P}_B)$ (resp. $f^{-1}(\mathbf{P}_{B^n})$) is the law of $f(B)$ (resp. $f(B^n)$) for

$f = \Gamma,\ \Psi_1,\ \Psi_2$. So if we concern the convergence in law, we can consider these equations on any probability space.

By Donsker's theorem and Skorokhod representation theorem, there exists a probability space, such that $\sup_{0\le t\le T}|B_t^n - B_t| \to 0$, as $n\to\infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_k is in $\mathbf{L}^{2+\delta}$. So we will work on this space with respect to the filtration generated by B^n and B, trying to prove the convergence of solutions. Thanks to the convergence of B^n, (L^n, U^n) also converges to (L, U). Then we have the following result, which is based on the convergence results of numerical solutions for BSDE (cf. [4], [5]) and penalization method for reflected BSDE (Theorem 2.1).

Proposition 3.1 *Assume 2.3 holds, the sequence* $(Y_t^{p,n}, Z_t^{p,n})$ *converges to* (Y_t, Z_t) *in following sense*

$$\lim_{p\to\infty}\lim_{n\to\infty} E[\sup_{0\le t\le T}|Y_t^{p,n} - Y_t|^2 + \int_0^T |Z_s^{p,n} - Z_s|^2\,ds] \to 0, \tag{12}$$

and for $0\le t\le T$, $A_t^{p,n} - K_t^{p,n} \to A_t - K_t$ *in* $\mathbf{L}^2(\mathcal{F}_t)$, *as* $n\to\infty$, $p\to\infty$.

Proof. Notice

$$\begin{aligned} E[\sup_{0\le t\le T}|Y_t^{p,n} - Y_t|^2 + \int_0^T |Z_s^{p,n} - Z_s|^2\,ds] &\le 2E[\sup_{0\le t\le T}|Y_t^{p,n} - Y_t^p|^2 + \int_0^T |Z_s^{p,n} - Z_s^p|^2\,ds] \\ &\quad +2E[\sup_{0\le t\le T}|Y_t^{p} - Y_t|^2 + \int_0^T |Z_s^{p} - Z_s|^2\,ds]. \end{aligned}$$

By the convergence results of numerical solutions for BSDE (cf. [4], [5]), the first part tends to 0. For the second part, it is a direct application of Theorem 2.1 of the penalization method. So we get (12). For the increasing processes, we have

$$\begin{aligned} E[((A_t^{p,n} - K_t^{p,n}) - (A_t - K_t))^2] &\le 2E[((A_t^{p,n} - K_t^{p,n}) - (A_t^p - K_t^p))^2] \\ &\quad +2E[((A_t^{p} - K_t^{p}) - (A_t - K_t))^2] \\ &\le 2E[((A_t^{p,n} - K_t^{p,n}) - (A_t^p - K_t^p))^2] + \frac{C}{\sqrt{p}}, \end{aligned}$$

in view of (7). While for fixed p,

$$\begin{aligned} A_t^{p,n} - K_t^{p,n} &= Y_0^{p,n} - Y_t^{p,n} - \int_0^t g(s, Y_s^{p,n}, Z_s^{p,n})ds + \int_0^t Z_s^{p,n} dB_s^n, \\ A_t^{p} - K_t^{p} &= Y_0^{p} - Y_t^{p} - \int_0^t g(s, Y_s^{p}, Z_s^{p})ds + \int_0^t Z_s^{p} dB_s, \end{aligned}$$

from Corollary 14 in [5], we know that $\int_0^\cdot Z_s^{p,n} dB_s^n$ converges to $\int_0^\cdot Z_s^p dB_s$ in $\mathbf{S}^2(0,T)$, as $n\to\infty$, then with the Lipschitz condition of g and the convergence of $Y^{p,n}$, we get $(A_t^{p,n} - K_t^{p,n}) \to (A_t - K_t)$ in $\mathbf{L}^2(\mathcal{F}_t)$, as $n\to\infty$, $p\to\infty$. $\square$

Now we consider the implicit–explicit penalization scheme. From Proposition 5 in [25], we know that for implicit–explicit scheme, the difference between this solution and the totally implicit one depends on $\mu + p$ for fixed $p\in\mathbb{N}$. So we have

Proposition 3.2 *For any $p \in \mathbb{N}$, when $n \to \infty$,*

$$E[\sup_{0\le t\le T} \left|\overline{Y}_t^{p,n} - Y_t^{p,n}\right|^2 + \int_0^T \left|\overline{Z}_s^{p,n} - Z_s^{p,n}\right|^2 ds \to 0,$$

with $(\overline{A}_t^{p,n} - \overline{K}_t^{p,n}) - (A_t^{p,n} - K_t^{p,n}) \to 0$ in $\mathbf{L}^2(\mathcal{F}_t)$, for $0 \le t \le T$.

Proof. The convergence of $(\overline{Y}_t^{p,n}, \overline{Z}_t^{p,n})$ to $(Y_t^{p,n}, Z_t^{p,n})$ is a direct consequence of Proposition 5 in [25]. More precisely, there exists a constant C which depends only on μ and T, such that

$$E[\sup_{0\le t\le T} \left|\overline{Y}_t^{p,n} - Y_t^{p,n}\right|^2] + E\int_0^T \left|\overline{Z}_s^{p,n} - Z_s^{p,n}\right|^2 ds \le C\delta^2.$$

Then we consider the convergence of the increasing processes, notice that for $0 \le t \le T$,

$$\overline{A}_t^{p,n} - \overline{K}_t^{p,n} = \overline{Y}_0^{p,n} - \overline{Y}_t^{p,n} - \int_0^t g(s, \overline{Y}_s^{p,n}, \overline{Z}_s^{p,n}) ds + \int_0^t \overline{Z}_s^{p,n} dB_s^n,$$

compare with $A_t^{p,n} - K_t^{p,n} = Y_0^{p,n} - Y_t^{p,n} - \int_0^t g(s, Y_s^{p,n}, Z_s^{p,n}) ds + \int_0^t Z_s^{p,n} dB_s^n$, thanks to the Lipschitz condition of g and the convergence of $(\overline{Y}^{p,n}, \overline{Z}^{p,n})$, we get $\overline{A}_t^{p,n} - \overline{K}_t^{p,n} \to A_t^{p,n} - K_t^{p,n}$, in $\mathbf{L}^2(\mathcal{F}_t)$, as $n \to \infty$, for fixed p. So the result follows. $\square$

Remark 3.1 *From this proposition and Proposition 3.1, we get the convergence of the implicit–explicit penalization scheme.*

Before going further, we prove an *a-priori* estimation of $(y^{p,n}, z^{p,n}, a^{p,n}, k^{p,n})$. This result will help us to get the convergence of reflected scheme, which will be discussed in the next section. Throughout this paper, we use $C_{\phi,\psi,\cdots}$ to denote a constant which depends on ϕ, ψ, $\cdots$. Here ϕ, ψ, $\cdots$ can be random variables or stochastic processes.

Lemma 3.2 *For each $p \in \mathbb{N}$ and δ such that $\delta(1 + 2\mu + 2\mu^2) < 1$, there exists a constant c such that*

$$E[\sup_j \left|y_j^{p,n}\right|^2 + \sum_{j=0}^n \left|z_j^{p,n}\right|^2 \delta + \frac{1}{p\delta}\sum_{j=0}^n \left|a_j^{p,n}\right|^2 + \frac{1}{p\delta}\sum_{j=0}^n \left|k_j^{p,n}\right|^2] \le c\, C_{\xi^n,g,L^n,U^n}.$$

Here C_{ξ^n,g,L^n,U^n} depends on ξ^n, $g(t,0,0)$, $(L^n)^+$ and $(U^n)^-$, while c depends only on μ and T.

Proof. Recall (9), we apply 'discrete Itô formula' (cf. [20]) for $(y_j^{p,n})^2$, we get

$$\begin{aligned} E[\left|y_j^{p,n}\right|^2 + \sum_{i=j}^{n-1} |z_i^{p,n}|^2 \delta] \quad &\le \quad E[|\xi^n|^2 + 2[\sum_{i=j}^{n-1} y_i^{p,n} \, |g(t_i, y_i^{p,n}, z_i^{p,n})| \, \delta] \\ &\quad +2E\sum_{i=j}^{n-1}(y_i^{p,n} \cdot a_i^{p,n} - y_i^{p,n} \cdot k_i^{p,n}). \end{aligned}$$

Since $y_i^{p,n} \cdot a_i^{p,n} = -p\delta((y_i^{p,n} - L_i^n)^-)^2 + p\delta L_i^n (y_i^{p,n} - L_i^n)^- = \frac{1}{p\delta} a_i^{p,n} + L_i^n a_i^{p,n}$ and $y_i^{p,n} \cdot k_i^{p,n} = p\delta((y_i^{p,n} - U_i^n)^+)^2 + U_i^n p\delta(y_i^{p,n} - U_i^n)^+ = \frac{1}{p\delta} k_i^{p,n} + U_i^n k_i^{p,n}$, we have

$$
\begin{aligned}
& E[\left|y_j^{p,n}\right|^2 + \frac{1}{2}\sum_{i=j}^{n-1} |z_i^{p,n}|^2 \, \delta] + 2E[\frac{1}{p\delta}\sum_{i=j}^{n-1}(a_i^{p,n})^2 + \frac{1}{p\delta}\sum_{i=j}^{n-1}(k_i^{p,n})^2] \\
\leq \;& E[|\xi^n|^2 + \sum_{i=j}^{n-1} |g(t_i,0,0)|^2 \, \delta + (1+2\mu+2\mu^2)\sum_{i=j}^{n-1} |y_i^{p,n}|^2 \, \delta + 2\sum_{i=j}^{n-1}(L_i^n)^+ a_i^{p,n} + 2\sum_{i=j}^{n-1}(U_i^n)^- k_i^{p,n}] \\
\leq \;& E[|\xi^n|^2 + \sum_{i=j}^{n-1} |g(t_i,0,0)|^2 \, \delta] + (1+2\mu+2\mu^2)\delta E\sum_{i=j}^{n-1} |y_i^{p,n}|^2 + \frac{1}{\alpha}E(\sum_{i=j}^{n-1} a_i^{p,n})^2 \\
& +\alpha E \sup_{j\leq i\leq n-1}((L_i^n)^+)^2 + \frac{1}{\beta}E(\sum_{i=j}^{n-1} k_i^{p,n})^2 + \beta E \sup_{j\leq i\leq n-1}((U_i^n)^+)^2.
\end{aligned}
$$

Since L^n and U^n are approximations of Itô processes, we can find a process X_j^n of the form $X_j^n = X_0 - \sum_{i=0}^{j-1} \sigma_i \varepsilon_{i+1}^n \sqrt{\delta} + V_j^{+n} - V_j^{-n}$, where $V_j^{\pm n}$ are $\mathcal{G}_j^n$-adapted increasing processes with $E[|V_n^{+n}|^2 + |V_n^{-n}|^2] < +\infty$, and $L_j^n \leq X_j^n \leq U_j^n$ holds. Then we apply similar techniques of stopping times as in the proof of Lemma 2 in [16] for the discrete case with $L_j^n \leq X_j^n \leq U_j^n$, we can prove

$$
E(\sum_{i=j}^{n-1} a_i^{p,n})^2 + E(\sum_{i=j}^{n-1} k_i^{p,n})^2 \leq 3\mu(C_{\xi^n,g,X^n} + E\sum_{i=j}^{n-1}[|y_i^{p,n}|^2 + |z_i^{p,n}|^2]\delta).
$$

While X^n can be controlled by L^n and U^n, we can replace it by L^n and U^n. Set $\alpha = \beta = 12\mu$ in the previous inequality, with Lemma 3.1, we get

$$
\sup_j E[\left|y_j^{p,n}\right|^2] + E[\sum_{i=0}^{n-1} |z_i^{p,n}|^2 \, \delta] + \frac{1}{p\delta}\sum_{i=0}^{n-1}(a_i^{p,n})^2 + \frac{1}{p\delta}\sum_{i=0}^{n-1}(k_i^{p,n})^2] \leq cC_{\xi^n,g,L^n,U^n}.
$$

We reconsider Itô formula for $|y_j^{p,n}|^2$, the take $\sup_j$ before expectation. Using Burkholder-Davis-Gundy inequality for martingale part $\sum_{i=0}^{j} y_j^{p,n} z_j^{p,n} \sqrt{\delta} \varepsilon_{j+1}^n$, with similar techniques, we get

$$
E[\sup_j \left|y_j^{p,n}\right|^2] \leq C_{\xi^n,g,L^n,U^n} + C_\mu E[\sum_{i=0}^{n-1} |y_i^{p,n}|^2 \, \delta] \leq C_{\xi^n,g,L^n,U^n} + C_\mu T \sup_j E[|y_j^{p,n}|^2].
$$

It follows the desired results. $\square$

4 Reflected Algorithms and their convergence

4.1 Reflected Schemes

This type of numerical schemes is based on reflecting the solution y^n between two barriers by a^n and k^n directly. In such a way the discrete solution y^n really stays between two barriers L^n

and U^n. After discretization of time interval, our discrete reflected BSDE with two barriers on small interval $[t_j, t_{j+1}]$, for $0 \le j \le n-1$, is

$$y_j^n = y_{j+1}^n + g(t_j, y_j^n, z_j^n)\delta + a_j^n - k_j^n - z_j^n \sqrt{\delta}\varepsilon_{j+1}^n, \tag{13}$$

with terminal condition $y_n^n = \xi^n$, and constraint and discrete integral conditions hold:

$$\begin{aligned} a_j^n &\ge 0,\ \ k_j^n \ge 0,\ \ a_j^n \cdot k_j^n = 0, \\ L_j^n &\le y_j^n \le U_j^n,\ \ (y_j^n - L_j^n)a_j^n = (y_j^n - U_j^n)k_j^n = 0. \end{aligned} \tag{14}$$

Note that, all terms in (13) are $\mathcal{G}_j^n$-measurable except y_{j+1}^n and ε_{j+1}^n.

The key point of our numerical schemes is how to solve $(y_j^n, z_j^n, a_j^n, k_j^n)$ from (13) using the $\mathcal{G}_{j+1}^n$-measurable random variable y_{j+1}^n obtained in the preceding step. First z_j^n is obtained by

$$z_j^n = E[y_{j+1}^n \varepsilon_{j+1}^n | \mathcal{G}_j^n] = \frac{1}{2\sqrt{\delta}}(y_{j+1}^n|_{\varepsilon_j^n = 1} - y_{j+1}^n|_{\varepsilon_j^n = -1}).$$

Then (13) with (14) becomes

$$\begin{aligned} y_j^n &= E[y_{j+1}^n|\mathcal{G}_j^n] + g(t_j, y_j^n, z_j^n)\delta + a_j^n - k_j^n,\ a_j^n \ge 0,\ k_j^n \ge 0, \\ L_j^n &\le y_j^n \le U_j^n,\ \ (y_j^n - L_j^n)a_j^n = (y_j^n - U_j^n)k_j^n = 0. \end{aligned} \tag{15}$$

Set $\Theta(y) := y - g(t_j, y, z_j^n)\delta$. In view of $\langle \Theta(y) - \Theta(y'), y - y' \rangle \ge (1 - \delta\mu)\left|y - y'\right|^2 > 0$, for δ small enough, we get that in such case $\Theta(y)$ is strictly increasing in y. So

$$\begin{aligned} y &\ge L_j^n \Longleftrightarrow \Theta(y) \ge \Theta(L_j^n), \\ y &\le U_j^n \Longleftrightarrow \Theta(y) \le \Theta(U_j^n). \end{aligned}$$

Then implicit reflected scheme gives the results with $E[y_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2}(y_{j+1}^n|_{\varepsilon_j^n=1} + y_{j+1}^n|_{\varepsilon_j^n=-1})$ as follows

$$\begin{aligned} y_j^n &= \Theta^{-1}(E[y_{j+1}^n|\mathcal{G}_j^n] + a_j^n - k_j^n), \\ a_j^n &= \left(E[y_{j+1}^n|\mathcal{G}_j^n] + g(t_j, L_j^n, z_j^n)\delta - L_j^n\right)^-, \\ k_j^n &= \left(E[y_{j+1}^n|\mathcal{G}_j^n] + g(t_j, U_j^n, z_j^n)\delta - U_j^n\right)^+, \end{aligned}$$

on the set $\{L_j^n < U_j^n\}$, then we know that $\{y_j^n - L_j^n = 0\}$ and $\{y_j^n - U_j^n = 0\}$ are disjoint. So with $(y_j^n - L_j^n)a_j^n = (y_j^n - U_j^n)k_j^n = 0$, we have $a_j^n \cdot k_j^n = 0$. On the set $\{L_j^n = U_j^n\}$, we get $a_j^n = (I_j^n)^+$ and $k_j^n = (I_j^n)^-$ by defintion, where $I_j^n := E[y_{j+1}^n|\mathcal{G}_j^n] + g(t_j, L_j^n, z_j^n)\delta - L_j^n$. So automatically $a_j^n \cdot k_j^n = 0$.

Our explicit reflected scheme is introduced by replacing y_j^n in g by $E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]$ in (15). So we get the following equation,

$$\begin{aligned} \bar{y}_j^n &= E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n,\ \overline{a}_j^n \ge 0,\ \overline{k}_j^n \ge 0, \\ L_j^n &\le \bar{y}_j^n \le U_j^n,\ \ (\bar{y}_j^n - L_j^n)\overline{a}_j^n = (\bar{y}_j^n - U_j^n)\overline{k}_j^n = 0. \end{aligned} \tag{16}$$

Then with $E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]=\frac{1}{2}(\overline{y}_{j+1}^n|_{\varepsilon_j^n=1}+\overline{y}_{j+1}^n|_{\varepsilon_j^n=-1})$, we get the solution

$$\begin{aligned}
\overline{y}_j^n &= E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]+g(t_j,E[\bar{y}_{j+1}^n|\mathcal{G}_j^n],\bar{z}_j^n)\delta+\overline{a}_j^n-\overline{k}_j^n, \\
\overline{a}_j^n &= \left(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]+g(t_j,E[\bar{y}_{j+1}^n|\mathcal{G}_j^n],\bar{z}_j^n)\delta-L_j^n\right)^-, \\
\overline{k}_j^n &= \left(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]+g(t_j,E[\bar{y}_{j+1}^n|\mathcal{G}_j^n],\bar{z}_j^n)\delta-U_j^n\right)^+.
\end{aligned} \tag{17}$$

4.2 Convergence of Reflected Implicit Schemes

Now we study the convergence of Reflected Schemes. For implicit reflected scheme, we denote

$$Y_t^n=y_{[t/\delta]}^n,\ Z_t^n=z_{[t/\delta]}^n,\ \ A_t^n=\sum_{i=0}^{[t/\delta]}a_i^n,\ \ K_t^n=\sum_{i=0}^{[t/\delta]}k_i^n,$$

and for explicit reflected scheme

$$\bar{Y}_t^n=\bar{y}_{[t/\delta]}^n,\ \bar{Z}_t^n=\bar{z}_{[t/\delta]}^n,\ \ \overline{A}_t^n=\sum_{i=0}^{[t/\delta]}\overline{a}_i^n,\ \ \bar{K}_t^n=\sum_{i=0}^{[t/\delta]}\overline{k}_i^n.$$

First we prove an estimation result for (y^n,z^n,a^n,k^n).

Lemma 4.1 *For δ such that $\delta(1+2\mu+2\mu^2)<1$, there exists a constant c depending only on μ and T such that*

$$E[\sup_j\left|y_j^n\right|^2+\sum_{j=0}^{n-1}\left|z_j^n\right|^2\delta+\left|\sum_{j=0}^{n-1}a_j^n\right|^2+\left|\sum_{j=0}^{n-1}k_j^n\right|^2]\leq c\,C_{\xi^n,g,L^n,U^n}.$$

Proof. First we consider the estimation of a_i^n and k_i^n. In view of $L_j^n\leq Y_j^n\leq U_j^n$, we have that

$$\begin{aligned}
a_j^n &\leq \left(E[L_{j+1}^n|\mathcal{G}_j^n]+g(t_j,L_j^n,z_j^n)\delta-L_j^n\right)^-=\delta(l_j+g(t_j,L_j^n,z_j^n))^-, \\
k_j^n &\leq \left(E[U_{j+1}^n|\mathcal{G}_j^n]+g(t_j,U_j^n,z_j^n)\delta-U_j^n\right)^+=\delta\left(u_j+g(t_j,U_j^n,z_j^n)\right)^+.
\end{aligned} \tag{18}$$

We consider following discrete BSDEs with $\widehat{y}_n^n=\widetilde{y}_n^n=\xi^n$,

$$\begin{aligned}
\widehat{y}_j^n &= \widehat{y}_{j+1}^n+[g(t_j,\widehat{y}_j^n,\widehat{z}_j^n)+(l_j)^-+g(t_j,L_j^n,\widehat{z}_j^n)^-]\delta-\widehat{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \\
\widetilde{y}_j^n &= \widetilde{y}_{j+1}^n+[g(t_j,\widetilde{y}_j^n,\widetilde{z}_j^n)-(u_j)^+-g(t_j,U_j^n,\widetilde{z}_j^n)^+]\delta-\widetilde{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n.
\end{aligned}$$

Thanks to discrete comparison theorem in [20], we have $\widetilde{y}_j^n\leq y_j^n\leq\widehat{y}_j^n$, so

$$E[\sup_j\left|y_j^n\right|^2]\leq\max\{E[\sup_j\left|\widetilde{y}_j^n\right|^2],E[\sup_j\left|\widehat{y}_j^n\right|^2]\}\leq c\,C_{\xi^n,g,L^n,U^n}. \tag{19}$$

The last inequality follows from estimations of discrete solution of classical BSDE $(\widehat{y}_j^n)^2$ and $(\widetilde{y}_j^n)^2$, which is obtained by Itô formulae and the discrete Gronwall inequality in Lemma 3.1. For z_j^n, we use 'discrete Itô formula' (cf. [20]) again for $(y_j^n)^2$, and get

$$
\begin{aligned}
E\left|y_j^n\right|^2+\sum_{i=j}^{n-1}\left|z_i^n\right|^2\delta &= E[\left|\xi^n\right|^2+2\sum_{i=j}^{n-1}y_i^n g(t_i,y_i^n,z_i^n)\delta+2\sum_{i=j}^{n-1}y_i^n a_i^n-2\sum_{i=j}^{n-1}y_i^n k_i^n] \\
&\leq E[\left|\xi^n\right|^2+\sum_{i=j}^{n-1}\left|g(t_i,0,0)\right|^2\delta+\delta(1+2\mu+2\mu^2)\sum_{i=j}^{n-1}\left|y_i^n\right|^2+\frac{1}{2}\sum_{i=j}^{n-1}\left|z_i^n\right|^2\delta] \\
&\quad +\alpha E[\sup_j((L_j^n)^+)^2+\sup_j((U_j^n)^-)^2]+\frac{1}{\alpha}E[(\sum_{j=i}^{n-1}a_j^n)^2+(\sum_{j=i}^{n-1}k_j^n)^2],
\end{aligned}
$$

using $(y_i^n-L_i^n)a_i^n=0$ and $(y_i^n-U_i^n)k_i^n=0$. And from (18), we have

$$
\begin{aligned}
E(\sum_{j=i}^{n-1}a_j^n)^2 &\leq 4\delta E\sum_{j=i}^{n-1}[(l_j)^2+g(t_i,0,0)^2+\mu\left|L_j^n\right|^2+\mu\left|z_j^n\right|^2], \\
E(\sum_{j=i}^{n-1}k_j^n)^2 &\leq 4\delta E\sum_{j=i}^{n-1}[(u_j)^2+g(t_i,0,0)^2+\mu\left|U_j^n\right|^2+\mu\left|z_j^n\right|^2].
\end{aligned}
$$

Set $\alpha=32\mu$, it follows

$$
\begin{aligned}
E[\left|y_j^n\right|^2+\frac{1}{4}\sum_{i=j}^{n-1}\left|z_i^n\right|^2\delta] &\leq E[\left|\xi^n\right|^2+(1+\frac{1}{8\mu^2})\sum_{i=j}^{n-1}\left|g(t_i,0,0)\right|^2\delta]+\delta(1+2\mu+2\mu^2)\sum_{i=j}^{n-1}\left|y_i^n\right|^2 \\
&\quad +32\mu^2E[\sup_j((L_j^n)^+)^2+\sup_j((U_j^n)^-)^2]+\frac{1}{8\mu^2}E\sum_{j=i}^{n-1}[(l_j)^2+(u_j)^2] \\
&\quad +\frac{1}{8}\delta E\sum_{j=i}^{n-1}[\left|L_j^n\right|^2+\left|U_j^n\right|^2]
\end{aligned}
$$

With (19), we obtain $\sum_{i=0}^{n-1}\left|z_i^n\right|^2\delta\leq cC_{\xi^n,g,L^n,U^n}$. Then applying these estimations to (18), we obtain desired results. □

With arguments similar to those precede Proposition 3.1, the laws of the solutions (Y,Z,A,K) and (Y^n,Z^n,A^n,K^n) or $(\bar{Y}^n,\bar{Z}^n,\overline{A}^n,\bar{K}^n)$ to reflected BSDE depend only on $(\mathbf{P}_B,\Gamma^{-1}(\mathbf{P}_B),g,\ \Psi_1^{-1}(\mathbf{P}_B),\Psi_2^{-1}(\mathbf{P}_B))$ and $(\mathbf{P}_{B^n},\Gamma^{-1}(\mathbf{P}_{B^n}),g,\Psi_1^{-1}(\mathbf{P}_{B^n}),\Psi_2^{-1}(\mathbf{P}_{B^n}))$ where $f^{-1}(\mathbf{P}_B)$ (resp. $f^{-1}(\mathbf{P}_{B^n})$) is the law of $f(B)$ (resp. $f(B^n)$) for $f=\Gamma,\ \Psi_1,\ \Psi_2$. So if we concern the convergence in law, we can consider these equations on any probability space.

From Donsker's theorem and Skorokhod representation theorem, there exists a probability space satisfying $\sup_{0\leq t\leq T}\left|B_t^n-B_t\right|\to 0$, as $n\to\infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_k is in $\mathbf{L}^{2+\delta}$. And it is sufficient for us to prove convergence results in this probability space. Our convergence result for the implicit reflected scheme is as follows:

Theorem 4.1 *Under Assumption 2.3 and suppose moreover that g satisfies Lipschitz condition (1), we have when $n \to +\infty$,*

$$E[\sup_t |Y_t^n - Y_t|^2] + E\int_0^T |Z_t^n - Z_t|^2 dt \to 0, \tag{20}$$

and $A_t^n - K_t^n \to A_t - K_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, for $0 \le t \le T$.

Proof. The proof is done in three steps.

In the first step, we consider the difference between discrete solutions of reflect implicit scheme and of penalization implicit scheme introduce in section 4.1 and section 3.1, respectively. More precisely, we will prove that for each p,

$$E[\sup_j |y_j^n - y_j^{p,n}|^2] + \delta E \sum_{j=0}^{n-1} |z_j^n - z_j^{p,n}|^2 \le c C_{\xi^n, g, L^n, U^n} \frac{1}{\sqrt{p}}. \tag{21}$$

Here c only depends on μ and T. From (9) and (13), applying 'discrete Itô formula' (cf. [20]) to $(y_j^n - y_j^{p,n})^2$, we get

$$\begin{aligned}
& E\left|y_j^n - y_j^{p,n}\right|^2 + \delta E \sum_{i=j}^{n-1} |z_i^n - z_i^{p,n}|^2 \\
= \; & 2E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})(g(t_i, y_i^n, z_i^n) - g(t_i, y_i^{p,n}, z_i^{p,n}))\delta] \\
& + 2E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})(a_i^n - a_i^{p,n})] - 2E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})(k_i^n - k_i^{p,n})]
\end{aligned}$$

From (14), we have

$$\begin{aligned}
(y_i^n - y_i^{p,n})(a_i^n - a_i^{p,n}) &= (y_i^n - L_i^n)a_i^n - (y_i^{p,n} - L_i^n)a_i^n - (y_i^n - L_i^n)a_i^{p,n} + (y_i^{p,n} - L_i^n)a_i^{p,n} \\
&\le (y_i^{p,n} - L_i^n)^- a_i^n - ((y_i^{p,n} - L_i^n)^-)^2, \\
&\le (y_i^{p,n} - L_i^n)^- a_i^n,
\end{aligned}$$

Similarly we have $(y_i^n - y_i^{p,n})(k_i^n - k_i^{p,n}) \ge -(y_i^{p,n} - U_i^n)k_i^n$. By (18) and the Lipschitz property of g, it follows

$$\begin{aligned}
& E\left|y_j^n - y_j^{p,n}\right|^2 + \frac{\delta}{2} E\sum_{i=j}^{n-1} |z_i^n - z_i^{p,n}|^2 \\
\le \; & (2\mu + 2\mu^2)\delta E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})^2] + 2E\sum_{i=j}^{n-1}[(y_i^{p,n} - L_i^n)^- a_i^n + (y_i^{p,n} - U_i^n)^+ k_i^n] \\
\le \; & (2\mu + 2\mu^2)\delta E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})^2] + 2\left(\delta E\sum_{i=j}^{n-1}((y_i^{p,n} - L_i^n)^-)^2\right)^{\frac{1}{2}} \left(\delta E\sum_{i=j}^{n-1}((l_j + g(t_i, L_j^n, z_j^n))^-)^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$+2\left(\delta E\sum_{i=j}^{n-1}((y_i^{p,n}-U_i^n)^+)^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((u_j+g(t_i,U_j^n,z_j^n))^+)^2\right)^{\frac{1}{2}}$$

$$= (2\mu+2\mu^2)\delta E\sum_{i=j}^{n-1}[(y_i^n-y_i^{p,n})^2]+\frac{2}{\sqrt{p}}\left(\frac{1}{p\delta}E\sum_{i=j}^{n-1}(a_i^{p,n})^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((l_j+g(t_i,L_j^n,z_j^n))^-)^2\right)^{\frac{1}{2}}$$

$$+\frac{2}{\sqrt{p}}\left(\frac{1}{p\delta}E\sum_{i=j}^{n-1}(k_i^{p,n})^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((u_j+g(t_i,U_j^n,z_j^n))^+)^2\right)^{\frac{1}{2}}.$$

Then by estimation results in Lemma 3.2, Lemma 4.1 and discrete Gronwall inequality in Lemma 3.1, we get

$$\sup_j E\left|y_j^n-y_j^{p,n}\right|^2+\delta E\sum_{i=0}^{n-1}|z_i^n-z_i^{p,n}|^2\le c\,C_{\xi^n,g,L^n,U^n}\frac{1}{\sqrt{p}}.$$

Apply B-D-G inequality, we obtain (21).

In the second step, we want to prove (20). We have

$$\begin{aligned}
& E[\sup_t|Y_t^n-Y_t|^2]+E[\int_0^T|Z_t^n-Z_t|^2dt] \\
\le\ & 3E[\sup_t|Y_t^p-Y_t|^2+\int_0^T|Z_t^p-Z_t|^2dt]+3E[\sup_t|Y_t^n-Y_t^{p,n}|^2+\int_0^T|Z_t^n-Z_t^{p,n}|^2dt] \\
& +3E[\sup_t|Y_t^p-Y_t^{p,n}|^2+\int_0^T|Z_t^p-Z_t^{p,n}|^2dt] \\
\le\ & 3Cp^{-\frac{1}{2}}+cC_{\xi^n,g,L^n,U^n}p^{-\frac{1}{2}}+3E[\sup_t|Y_t^p-Y_t^{p,n}|^2+\int_0^T|Z_t^p-Z_t^{p,n}|^2dt],
\end{aligned}$$

in view of (21) and Theorem 2.1. For fixed $p>0$, by convergence results of numerical algorithms for BSDE, (Theorem 12 in [5] and Theorem 2 in [25]), we know that the last two terms converge to 0, as $\delta\to 0$. And when δ is small enough, C_{ξ^n,g,L^n,U^n} is dominated by ξ^n, g, L and U. This implies that we can choose suitable δ such that the right hand side is as small as we want, so (20) follows.

In the last step, we consider the convergence of (A^n,K^n). Recall that for $0\le t\le T$,

$$\begin{aligned}
A_t^n-K_t^n &= Y_0^n-Y_t^n-\int_0^t g(s,Y_s^n,Z_s^n)ds+\int_0^t Z_s^n dB_s^n, \\
A_t^{p,n}-K_t^{p,n} &= Y_0^{p,n}-Y_t^{p,n}-\int_0^t g(s,Y_s^{p,n},Z_s^{p,n})ds+\int_0^t Z_s^{p,n}dB_s^n.
\end{aligned}$$

By (21) and Lipschitz condition of g, we get

$$E[|(A_t^n-K_t^n)-(A_t^{p,n}-K_t^{p,n})|^2]\le c\,C_{\xi^n,g,L^n,U^n}\frac{1}{\sqrt{p}}.$$

Since

$$\begin{aligned} E[|(A_t^n - K_t^n) - (A_t - K_t)|^2] &\le 3E[|(A_t^n - K_t^n) - (A_t^{p,n} - K_t^{p,n})|^2] + 3E[|(A_t^p - K_t^p) - (A_t - K_t)|^2] \\ &\quad +3E[|(A_t^p - K_t^p) - (A_t^{p,n} - K_t^{p,n})|^2] \\ &\le c(C_{\xi^n,g,L^n,U^n} + C_{\xi,g,L,U})\frac{1}{\sqrt{p}} + 3E[|(A_t^p - K_t^p) - (A_t^{p,n} - K_t^{p,n})|^2], \end{aligned}$$

with similar techniques, we obtain $E[|(A_t^n - K_t^n) - (A_t - K_t)|^2] \to 0$. Here the fact that $(A_t^{p,n} - K_t^{p,n})$ converges to $(A_t^p - K_t^p)$ for fixed p follows from the convergence results of $(Y_t^{p,n}, Z_t^{p,n})$ to (Y_t^p, Z_t^p). $\square$

4.3 Convergence of Reflected Explicit Scheme

Then we study the convergence of explicit reflected scheme. Before going further, we need an estimation result for $(\overline{y}^n, \overline{z}^n, \overline{a}^n, \overline{k}^n)$.

Lemma 4.2 *For δ such that $\delta(\frac{9}{4} + 2\mu + 4\mu^2) < 1$, there exists a constant c depending only on μ and T, such that*

$$E[\sup_j |\overline{y}_j^n|^2] + E[\sum_{j=0}^{n-1} |\overline{z}_j^n|^2 \delta + \left|\sum_{j=0}^{n-1} \overline{k}_j^n\right|^2 + \left|\sum_{j=0}^{n-1} \overline{a}_j^n\right|^2] \le c\, C_{\xi^n,g,L^n,U^n}.$$

Proof. We recall the explicit reflected scheme, which is:

$$\begin{aligned} \bar{y}_j^n &= \bar{y}_{j+1}^n + g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n - \bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n,\ \overline{a}_j^n \ge 0,\ \overline{k}_j^n \ge 0, \\ L_j^n &\le \bar{y}_j^n \le U_j^n,\ \ (\bar{y}_j^n - L_j^n)\overline{a}_j^n = (\bar{y}_j^n - U_j^n)\overline{k}_j^n = 0. \end{aligned} \tag{22}$$

Then we have

$$\begin{aligned} |\bar{y}_j^n|^2 &= |\overline{y}_{j+1}^n|^2 - |\bar{z}_j^n|^2\delta + 2\bar{y}_{j+1}^n \cdot g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\delta + 2\bar{y}_j^n \cdot \overline{a}_j^n - 2\bar{y}_j^n \cdot \overline{k}_j^n \\ &\quad +|g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)|^2\delta^2 - (\overline{a}_j^n)^2 - (\overline{k}_j^n)^2 - 2\bar{y}_j^n\bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n \\ &\quad +2g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\bar{z}_j^n\delta\sqrt{\delta}\varepsilon_{j+1}^n - 2(\overline{a}_j^n - \overline{k}_j^n)\bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n. \end{aligned} \tag{23}$$

In view of $(\overline{y}_j^n - L_j^n)\overline{a}_j^n = (\bar{y}_j^n - U_j^n)\overline{k}_j^n = 0$, $\overline{a}_j^n$ and $\overline{k}_j^n \ge 0$, and taking expectation, we have

$$\begin{aligned} E|\bar{y}_j^n|^2 + E|\bar{z}_j^n|^2\delta &\le E|\overline{y}_{j+1}^n|^2 + 2E[\overline{y}_{j+1}^n \cdot g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)]\delta + 2E[(L_j^n)^+ \cdot \overline{a}_j^n] + E[(U_j^n)^- \cdot \overline{k}_j^n] \\ &\quad +E[|g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)|^2\delta^2] \\ &\le E|\overline{y}_{j+1}^n|^2 + (\delta + 3\delta^2)E[|g(t_j,0,0)|^2] + (\frac{1}{4}\delta + 3\mu^2\delta^2)E[(\overline{z}_j^n)^2] \\ &\quad +\delta(1 + 2\mu + 4\mu^2 + 3\mu^2\delta)E|\overline{y}_{j+1}^n|^2 + 2E[(L_j^n)^+ \cdot \overline{a}_j^n] + E[(U_j^n)^- \cdot \overline{k}_j^n] \end{aligned}$$

Taking the sum for $j = i, \cdots, n-1$ yields

$$E|\bar{y}_i^n|^2 + \frac{1}{2}\sum_{j=i}^{n-1} E|\bar{z}_j^n|^2\delta \tag{24}$$
$$\leq E|\xi^n|^2 + (\delta + 3\delta^2)E\sum_{j=i}^{n-1}[|g(t_j,0,0)|^2] + \delta(1 + 2\mu + 4\mu^2 + 3\mu^2\delta)E\sum_{j=i}^{n-1}|\overline{y}_{j+1}^n|^2$$
$$+\alpha E[\sup_j((L_j^n)^+)^2 + \sup_j((U_j^n)^+)^2] + \frac{1}{\alpha}E[(\sum_{j=i}^{n-1}\overline{a}_j^n)^2 + (\sum_{j=i}^{n-1}\overline{k}_j^n)^2],$$

where α is a constant to be decided later. From (17), we have

$$\begin{aligned}
\overline{a}_j^n &\leq \left(E[L_{j+1}^n|\mathcal{G}_j^n] + g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta - L_j^n\right)^- = (l_j + g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n))^-\delta, \\
\overline{k}_j^n &\leq \left(E[U_{j+1}^n|\mathcal{G}_j^n] + g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta - U_j^n\right)^+ = (u_j + g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n))^-\delta.
\end{aligned}$$

Then we get

$$\begin{aligned}
E(\sum_{j=i}^{n-1}\overline{a}_j^n)^2 &\leq 4\delta E\sum_{j=i}^{n-1}[(l_j)^2 + g(t_j,0,0)^2 + \mu^2(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n])^2 + \mu^2(\bar{z}_j^n)^2], \qquad (25)\\
E(\sum_{j=i}^{n-1}\overline{k}_j^n)^2 &\leq 4\delta E\sum_{j=i}^{n-1}[(u_j)^2 + g(t_j,0,0)^2 + \mu^2(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n])^2 + \mu^2(\bar{z}_j^n)^2],
\end{aligned}$$

Set $\alpha = 32\mu^2$ in (24), it follows

$$E|\bar{y}_i^n|^2 + \frac{1}{4}\sum_{j=i}^{n-1}E|\bar{z}_j^n|^2\delta$$
$$\leq E|\xi^n|^2 + (\delta + \frac{1}{4\mu^2}\delta + 3\delta^2)E\sum_{j=i}^{n-1}[|g(t_j,0,0)|^2] + 32\mu^2 E[\sup_j((L_j^n)^+)^2 + \sup_j((U_j^n)^+)^2]$$
$$+\delta(\frac{5}{4} + 2\mu + 4\mu^2 + 3\mu^2\delta)E\sum_{j=i}^{n-1}|\overline{y}_{j+1}^n|^2 + \frac{1}{8\mu^2}\delta E\sum_{j=i}^{n-1}[(u_i^l)^2 + (u_i^u)^2].$$

Notice that $3\mu^2\delta < 1$, so $3\mu^2\delta^2 < \delta$. Then by applying the discrete Gronwall inequality in Lemma 3.1, and the estimation of $\overline{a}_j^n$ and $\overline{k}_j^n$ follows from (25) , we get

$$\sup_j E[\left|\overline{y}_j^n\right|^2] + E[\sum_{j=0}^{n-1}\left|\overline{z}_j^n\right|^2\delta + \left|\sum_{j=0}^{n-1}\overline{k}_j^n\right|^2 + \left|\sum_{j=0}^{n-1}\overline{a}_j^n\right|^2] \leq c\, C_{\xi^n,g,L^n,U^n}.$$

We reconsider (23), as before take sum and $\sup_j$, then take expectation, using Burkholder-Davis-Gundy inequality for martingale part, with similar techniques, we get

$$E[\sup_j\left|\overline{y}_j^n\right|^2] \leq C_{\xi^n,g,L^n,U^n} + C_\mu E[\sum_{j=0}^{n-1}\left|\overline{y}_j^n\right|^2\delta \leq E[\sup_j\left|\overline{y}_j^n\right|^2] \leq C_{\xi^n,g,L^n,U^n} + C_\mu T\sup_j E[\left|\overline{y}_j^n\right|^2],$$

which implies final result. $\square$

Then our convergence result for the explicit reflected scheme is

Theorem 4.2 *Under the same assumptions as in Theorem 4.1, when $n \to +\infty$,*

$$E[\sup_t |\overline{Y}_t^n - Y_t|^2] + E\int_0^T |\overline{Z}_t^n - Z_t|^2 dt \to 0. \tag{26}$$

And $\overline{A}_t^n - \overline{K}_t^n \to A_t - K_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, for $0 \le t \le T$.

Proof. Thanks to Theorem 4.1, it is sufficient to prove that as $n \to +\infty$,

$$E[\sup_j |\overline{y}_j^n - y_j^n|^2] + E\sum_{j=0}^{n-1} |\overline{z}_j^n - z_j^n|^2 \delta \to 0. \tag{27}$$

Since

$$\begin{aligned} y_j^n &= y_{j+1}^n + g(t_j, y_j^n, z_j^n)\delta + a_j^n - k_j^n - z_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \\ \overline{y}_j^n &= E[\overline{y}_{j+1}^n|\mathcal{G}_j^n] + g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n - \overline{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \end{aligned} \tag{28}$$

we get

$$\begin{aligned} & E\left|y_j^n - \overline{y}_j^n\right|^2 \\ = \; & E\left|y_{j+1}^n - \overline{y}_{j+1}^n\right|^2 - \delta E\left|z_j^n - \overline{z}_j^n\right|^2 + 2\delta E[(y_j^n - \overline{y}_j^n)(g(t_j, y_j^n, z_j^n) - g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n))] \\ & -E[\delta(g(t_j, y_j^n, z_j^n) - g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n)) + (a_j^n - \overline{a}_j^n) - (k_j^n - \overline{k}_j^n)]^2 \\ & +2E[(y_j^n - \overline{y}_j^n)(a_j^n - \overline{a}_j^n)] - 2E[(y_j^n - \overline{y}_j^n)(k_j^n - \overline{k}_j^n)] \\ \le \; & E\left|y_{j+1}^n - \overline{y}_{j+1}^n\right|^2 - \delta E\left|z_j^n - \overline{z}_j^n\right|^2 + 2\delta E[(y_j^n - \overline{y}_j^n)(g(t_j, y_j^n, z_j^n) - g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n))] \end{aligned}$$

in view of

$$\begin{aligned} (y_j^n - \overline{y}_j^n)(a_j^n - \overline{a}_j^n) &= (y_j^n - L_j^n)a_j^n + (\overline{y}_j^n - L_j^n)(\overline{a}_j^n) \\ &\quad -(\overline{y}_j^n - L_j^n)a_j^n - (y_j^n - L_j^n)(\overline{a}_j^n) \\ &\le 0 \\ (y_j^n - \overline{y}_j^n)(k_j^n - \overline{k}_j^n) &= (y_j^n - U_j^n)k_j^n + (\overline{y}_j^n - U_j^n)\overline{k}_j^n \\ &\quad -(y_j^n - U_j^n)\overline{k}_j^n - (\overline{y}_j^n - U_j^n)k_j^n \\ &\ge 0. \end{aligned}$$

We take sum over j from i to $n-1$, with $\xi^n - \overline{\xi}^n = 0$, then we get

$$\begin{aligned} E\left|y_j^n - \overline{y}_j^n\right|^2 + \delta\sum_{j=i}^{n-1} E\left|z_j^n - \overline{z}_j^n\right|^2 &\le 2\delta\sum_{j=i}^{n-1} E[(y_j^n - \overline{y}_j^n)(g(t_j, y_j^n, z_j^n) - g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n], \overline{z}_j^n))] \\ &\le 2\mu^2\delta E\sum_{j=i}^{n-1}\left|y_j^n - \overline{y}_j^n\right|^2 + \frac{\delta}{2}\sum_{j=i}^{n-1} E\left|z_j^n - \overline{z}_j^n\right|^2 \\ &\quad +2\mu\delta E\sum_{j=i}^{n-1}\left|y_j^n - \overline{y}_j^n\right| \cdot \left|y_j^n - E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]\right|. \end{aligned}$$

Since $\bar{y}_j^n - E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] = g(t_j, E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n$, we have

$$\begin{aligned}
& 2\mu\delta E[\left|y_j^n - \overline{y}_j^n\right| \cdot \left|y_j^n - E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]\right|] \\
= \; & 2\mu\delta E[\left|y_j^n - \overline{y}_j^n\right| \cdot \left|y_j^n - \overline{y}_j^n + g(t_j, E[\overline{y}_{j+1}^n|\mathcal{G}_j^n]), z_j^n)\delta + \overline{a}_j^n - \overline{k}_j^n\right|] \\
\leq \; & (2\mu+1)\delta E[\left|y_j^n - \overline{y}_j^n\right|^2] + \mu^2\delta E[3\delta^2(\left|g(t_j,0,0)\right|^2 + \mu^2\left|\overline{y}_{j+1}^n\right|^2 + \mu^2\left|z_j^n\right|^2 + (\overline{a}_j^n)^2 + (\overline{k}_j^n)^2].
\end{aligned}$$

Then by Lemma 4.2, we obtain

$$E\left|y_j^n - \overline{y}_j^n\right|^2 + \frac{\delta}{2}\sum_{j=i}^{n-1} E\left|z_j^n - \overline{z}_j^n\right|^2 \leq (2\mu^2+2\mu+1)\delta\sum_{j=i}^{n-1} E\left|y_j^n - \overline{y}_j^n\right|^2 + \delta C_{\xi^n,g,L^n,U^n}. \quad (29)$$

By the discrete Gronwall inequality in Lemma 3.1, we get

$$\sup_{j\leq n} E\left|y_j^n - \overline{y}_j^n\right|^2 \leq C\delta^2 e^{(2\mu+2\mu^2+1)T}.$$

With (29), it follows $E[\delta\sum_{j=0}^{n-1} E\left|z_j^n - \overline{z}_j^n\right|^2] \leq C\delta^2$. Then we reconsider (28), this time we take expectation after taking square, sum and sup over j. Using Burkholder-Davis-Gundy inequality for martingale parts and similar tachniques, it follows that

$$E\sup_{j\leq n}\left|y_j^n - \overline{y}_j^n\right|^2 \leq CE[\sum_{j=0}^{n-1} E\left|y_j^n - \overline{y}_j^n\right|^2\delta] \leq CT\sup_{j\leq n} E\left|y_j^n - \overline{y}_j^n\right|^2,$$

which implies (27).

For the convergence of $(\overline{A}^n, \overline{K}^n)$, we consider

$$\begin{aligned}
\overline{A}_t^n - \overline{K}_t^n &= \overline{Y}_0^n - \overline{Y}_t^n - \int_0^t g(s, \overline{Y}_s^n, \overline{Z}_s^n)ds + \int_0^t \overline{Z}_s^n dB_s^n, \\
A_t^n - K_t^n &= Y_0^n - Y_t^n - \int_0^t g(s, Y_s^n, Z_s^n)ds + \int_0^t Z_s^n dB_s^n,
\end{aligned}$$

then the convergence results follow easily from the convergence of A^n, (26) and the Lipschitz condition of g. □

5 Simulations of Reflected BSDE with two barriers

For computational convenience, we consider the case when $T = 1$. The calculation begins from $y_n^n = \xi^n$ and proceeds backward to solve $(y_j^n, z_j^n, a_j^n, k_j^n)$, for $j = n-1, \cdots 1, 0$. Due to the amount of computation, we consider a very simple case: $\xi = \Phi(B_1)$, $L_t = \psi_1(t, B(t))$, $U_t = \psi_2(t, B(t))$, where Φ, ψ_1 and ψ_2 are real analytic functions defined on $\mathbb{R}$ and $[0,1]\times\mathbb{R}$ respectively. As mentioned in the introduction, we have developed a Matlab toolbox for calculating and simulating solutions of reflected BSDEs with two barriers which has a well-designed interface. This toolbox can be downloaded from http://159.226.47.50:8080/iam/xumingyu/English/index.jsp, with clicking 'Preprint' on the left side.

We take the following example: $g(y,z) = -5\,|y+z| - 1$, $\Phi(x) = |x|$, $\Psi_1(t,x) = -3(x-2)^2 + 3$, $\Psi_2(t,x) = (x+1)^2 + 3(t-1)$, and $n = 400$.

In Figure 1, we can see both the global situation of the solution surface of y^n and its partial situation i.e. trajectory. In the upper portion of Figure 1, it is in 3-dimensional. The lower surface shows the barrier L, as well the upper one is for the barrier U. The solution y^n is in the middle of them. Then we generate one trajectory of the discrete Brownian motion $(B_j^n)_{0\le j\le n}$, which is drawn on the horizontal plane. The value of y_j^n with respect to this Brownian sample is showed on the solution surface. The remainder of the figure shows respectively the trajectory of the force $A_j^n = \sum_{i=0}^{j} a_i^n$ and $K_j^n = \sum_{i=0}^{j} k_i^n$.

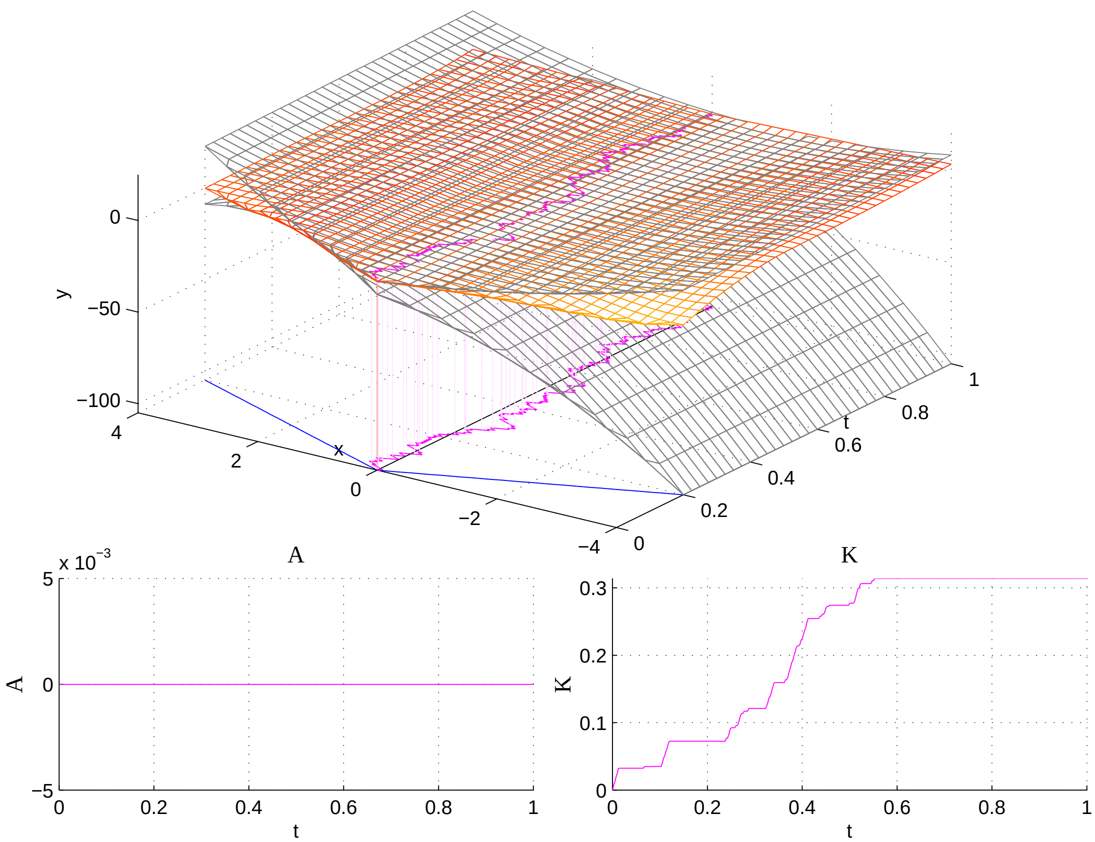

Figure 1: A solution surface of reflected BSDE with two barriers

The lower graphs shows clearly that A^n (respective K^n) acts only if y^n touches the lower barrier L^n, i.e. on the set $\{y^n = L^n\}$ (respective the upper barrier U, i.e. on the set $\{y^n = U^n\}$), and they never act at the same time.

In the upper portion we can see that there is an area, named Area I, (resp. Area II) where the solution surface and the lower barrier surface (resp. the solution surface and the upper barrier surface) stick together. When the trajectory of solution y_j^n goes into Area I (resp. Area II), the force A_j^n (resp. K_j^n) will push y_j^n upward (resp. downward). Indeed, if we don't have the barriers here, y_j^n intends going up or down to cross the reflecting barrier L_j^n and U_j^n, so to keep y_j^n staying between L_j^n and U_j^n, the action of forces A_j^n and K_j^n are necessary. In Figure 1, the increasing process A_j^n keeps zero, while K_j^n increases from the

beginning. Correspondingly in the beginning y_j^n goes into Area II, but always stay out of Area I. Since Area I and Area II are totally disjoint, so A_j^n and K_j^n never increase at same time.

About this point, let us have a look at Figure 2. This figure shows a group of 3-dimensional dynamic trajectories (t_j, B_j^n, Y_j^n) and (t_j, B_j^n, Z_j^n), simultaneously, of 2-dimensional trajectories of (t_j, Y_j^n) and (t_j, Z_j^n). For the other sub-figures, the upper-right one is for the trajectories A_j^n, and while the lower-left one is for K_j^n, then comparing these two sub-figures, as in Figure 1, $\{a_j^n \neq 0\}$ and $\{k_j^n \neq 0\}$ are disjoint, but the converse is not true.

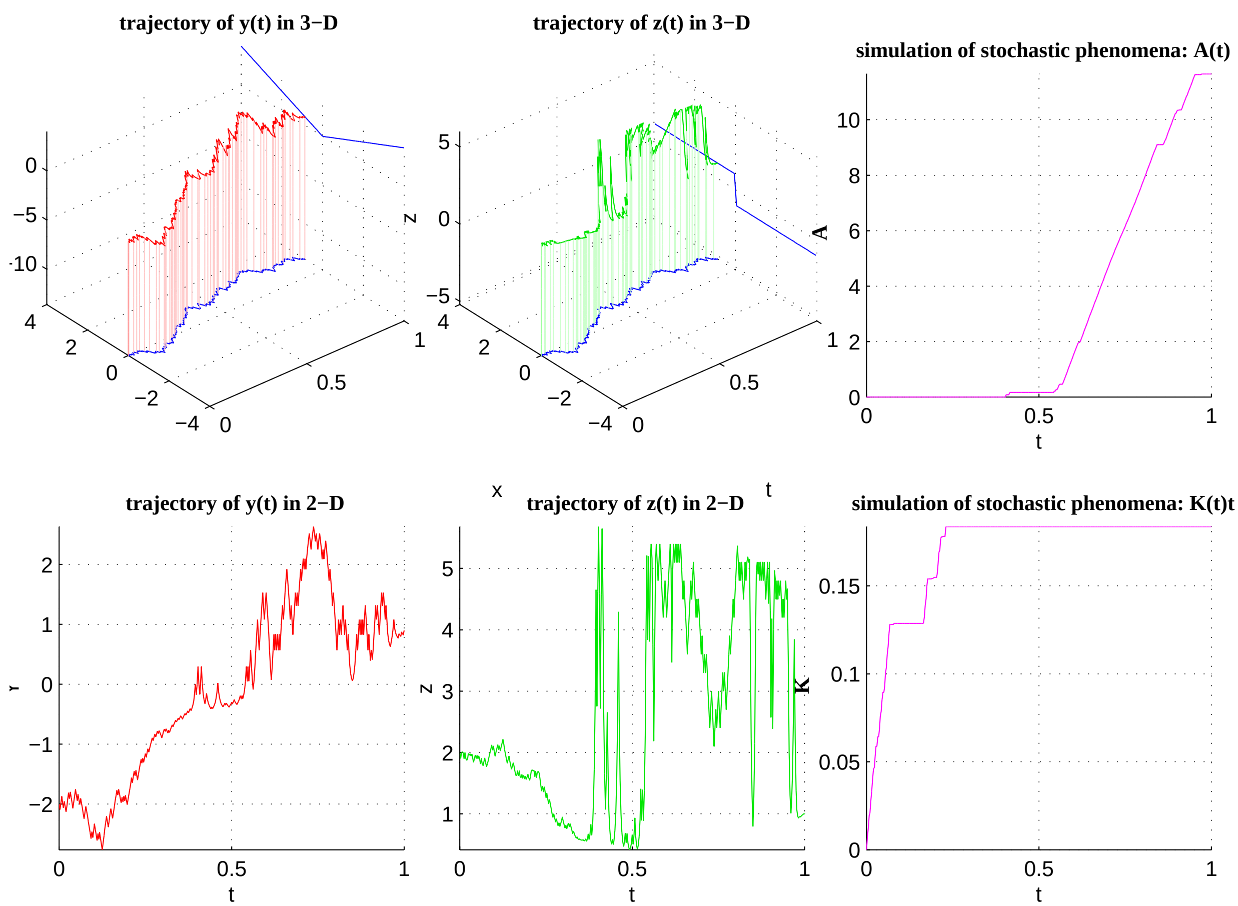

Figure 2: The trajectories of solutions of (3)

Now we present some numerical results using the explicit reflected scheme and the implicit-explicit penalization scheme, respectively, with different discretization. Consider the parameters: $g(y,z) = -5\,|y+z| - 1$, $\Phi(x) = |x|$, $\Psi_1(t,x) = -3(x-2)^2 + 3$, $\Psi_2(t,x) = (x+1)^2 + 3t - 2.5$:

$n = 400$, reflected explicit scheme: $y_0^n = -1.7312$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-1.8346	-1.7476	-1.7329	-1.7314

$n = 1000$, reflected explicit scheme: $y_0^n = -1.7142$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-1.8177	-1.7306	-1.7161	-1.7144

$n = 2000$, reflected explicit scheme: $y_0^n = -1.7084$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-1.8124	-1.7250	-1.7103	-1.7068

$n = 4000$, reflected explicit scheme: $y_0^n = -1.7055$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	-1.8096	-1.7222	-1.7074	-1.7057

From this form, first we can see that as the penalization parameter p increases, the penalization solution $y_0^{p,n}$ tends increasingly to the reflected solution y_0^n. Second, as the discretaization parameter n increases, the differences of y_0^n with different n become smaller as well as that of $y_0^{p,n}$.

6 Appendix: The proof of Theorem 2.1

To complete the paper, here we give the proof of Theorem 2.1.

Proof of Theorem 2.1. (a) is the main result in [16]. So we omit its proof.

Now we consider (b). The convergence of (Y_t^p, Z_t^p) is a direct consequence of [23]. For the convergence speed, the proof is a combination of results in [16] and [23]. From [16], we know that for (5), as $m \to \infty$, $\widehat{Y}_t^{m,p} \nearrow \underline{Y}_t^p$ in $\mathbf{S}^2(0,T)$, $\widehat{Z}_t^{m,p} \to \underline{Z}_t^p$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, $\widehat{A}_t^{m,p} \to \underline{A}_t^p$ in $\mathbf{S}^2(0,T)$, where $(\underline{Y}_t^p, \underline{Z}_t^p, \underline{A}_t^p)$ is a solution of the following reflected BSDE with one lower barrier L

$$\begin{aligned} -d\underline{Y}_t^p &= g(t, \underline{Y}_t^p, \underline{Z}_t^p)dt + d\underline{A}_t^p - p(\underline{Y}_t^p - U_t)^+ dt - \underline{Z}_t^p dB_t, \underline{Y}_T^p = \xi, \\ \underline{Y}_t^p &\geq L_t, \int_0^T (\underline{Y}_t^p - L_t) d\underline{A}_t^p = 0. \end{aligned} \tag{30}$$

Set $\underline{K}_t^p = \int_0^t p(\underline{Y}_s^p - U_s)^+ ds$. Then letting $p \to \infty$, it follows that $\underline{Y}_t^p \searrow Y_t$ in $\mathbf{S}^2(0,T)$, $\underline{Z}_t^p \to Z_t$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$. By comparison theorem, $d\underline{A}_t^p$ is increasing in p. So $\underline{A}_T^p \nearrow A_T$, and $0 \leq \sup_t [\underline{A}_t^{p+1} - \underline{A}_t^p] \leq \underline{A}_T^{p+1} - \underline{A}_T^p$. It follows that $\underline{A}_t^p \to A_t$ in $\mathbf{S}^2(0,T)$. Then with Lipschitz condition of g and convergence results, we get $\underline{K}_t^p \to K_t$ in $\mathbf{S}^2(0,T)$. Moreover from Lemma 4 in [16], we know that there exists a constant C depending on ξ, $g(t,0,0)$, μ, L and U, such that

$$E[\sup_{0 \leq t \leq T} |\underline{Y}_t^p - Y_t|^2 + \int_0^T (|\underline{Z}_t^p - Z_t|^2) dt \leq \frac{C}{\sqrt{p}}.$$

Similarly for (5), first letting $p \to \infty$, we get $\widehat{Y}_t^{m,p} \searrow \overline{Y}_t^m$ in $\mathbf{S}^2(0,T)$, $\widehat{Z}_t^{m,p} \to \overline{Z}_t^m$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, $\widehat{K}_t^{m,p} \to \overline{K}_t^m$ in $\mathbf{S}^2(0,T)$, where $(\overline{Y}_t^m, \overline{Z}_t^m, \overline{K}_t^m)$ is a solution of the following reflected BSDE

with one upper barrier U

$$
\begin{aligned}
-d\overline{Y}_t^m &= g(t,\overline{Y}_t^m,\overline{Z}_t^m)dt + m(L_t-\overline{Y}_t^m)^+dt - d\overline{K}_t^m - \overline{Z}_t^m dB_t, \overline{Y}_T^m = \xi, \qquad (31)\\
\overline{Y}_t^m &\leq U_t, \int_0^T (\overline{Y}_t^p - U_t)d\overline{K}_t^m = 0.
\end{aligned}
$$

In the same way, as $m\to\infty$, $\overline{Y}_t^m \nearrow Y_t$ in $\mathbf{S}^2(0,T)$, $\overline{Z}_t^m \to Z_t$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, and $(\overline{A}_t^m,\overline{K}_t^m)\to(A_t,K_t)$ in $(\mathbf{S}^2(0,T))^2$, where $\overline{A}_t^m = \int_0^t m(L_s-\overline{Y}_s^m)^+ds$. Also there exists a constant C depending on ξ, $g(t,0,0)$, μ, L and U, such that

$$
E[\sup_{0\leq t\leq T}|\overline{Y}_t^m - Y_t|^2 + \int_0^T(|\overline{Z}_t^m - Z_t|^2)dt \leq \frac{C}{\sqrt{m}}.
$$

Applying comparison theorem to (6) and (30), (6) and (31)(let $m=p$), we have $\overline{Y}_t^p \leq Y_t^p \leq \underline{Y}_t^p$. Then we have

$$
E[\sup_{0\leq t\leq T}|Y_t^p - Y_t|^2] \leq \frac{C}{\sqrt{p}},
$$

for some constant C. To get the estimate results for Z^p, we apply Itô formula to $|Y_t^p - Y_t|^2$, and get

$$
\begin{aligned}
& E|Y_0^p - Y_0|^2 + \frac{1}{2}E\int_0^T |Z_s^p - Z_s|^2 ds \\
= \;& (\mu+2\mu^2)E\int_0^T |Y_s^p - Y_s|^2\,ds + 2E\int_0^T (Y_s^p - Y_s)dA_s^p - 2E\int_0^T (Y_s^p - Y_s)dA_s \\
& -2E\int_0^T (Y_s^p - Y_s)dK_s^p + 2E\int_0^T (Y_s^p - Y_s)dK_s.
\end{aligned}
$$

Since

$$
\begin{aligned}
2E\int_0^T (Y_s^p - Y_s)dA_s^p &= 2E\int_0^T (Y_s^p - L_s)dA_s^p - 2E\int_0^T (Y_s - L_s)dA_s^p \\
&\leq 2pE\int_0^T (Y_s^p - L_s)(Y_s^p - L_s)^- ds \leq 0
\end{aligned}
$$

and $2E\int_0^T (Y_s^p - Y_s)dK_s^p \geq 2pE\int_0^T (Y_s^p - U_s)(Y_s^p - U_s)^+ds \geq 0$, we have

$$
E\int_0^T |Z_s^p - Z_s|^2 ds \leq \frac{C}{\sqrt{p}},
$$

in view of the estimation of A and K and the convergence of Y^p.

Now we consider the convergence of A^p and K^p. Since

$$
\begin{aligned}
A_t - K_t &= Y_0 - Y_t - \int_0^t g(s,Y_s,Z_s)ds + \int_0^t Z_s dB_s, \\
A_t^p - K_t^p &= Y_0^p - Y_t^p - \int_0^t g(s,Y_s^p,Z_s^p)ds + \int_0^t Z_s^p dB_s,
\end{aligned}
$$

from the Lipschitz condition of g and the convergence results of Y^p and Z^p, we have immediately

$$
\begin{aligned}
& E[\sup_{0\le t\le T}[(A_t-K_t)-(A_t^p-K_t^p)]^2] \\
\le\ & 8E[\sup_{0\le t\le T}|Y_t^p-Y_t|^2+4\mu\int_0^T|Y_s^p-Y_s|^2ds+C\int_0^T|Z_s^p-Z_s|^2ds]\le\frac{C}{\sqrt{p}}.
\end{aligned}
$$

Meanwhile we know $E[(A_T^p)^2+(K_T^p)^2]<\infty$, so A^p and K^p admits weak limit $\widetilde{A}$ and $\widetilde{K}$ in $\mathbf{S}^2(0,T)$ respectively. By the comparison results of $\overline{Y}_t^p$, Y_t^p and $\underline{Y}_t^p$, we get

$$
\begin{aligned}
dA_t^p &= p(Y_t^p-L_t)^-dt\le p(\overline{Y}_t^p-L_t)^-dt=d\overline{A}_t^p,\\
dK_t^p &= p(Y_t^p-U_t)^+dt\ge p(\underline{Y}_t^p-U_t)^+dt=d\underline{K}_t^p.
\end{aligned}
$$

So $d\widetilde{A}_t\le dA_t$ and $d\widetilde{K}_t\ge dK_t$, it follows that $d\widetilde{A}_t-d\widetilde{K}_t\le dA_t-dK_t$. On the other hand, the limit of Y^p is Y, so $d\widetilde{A}_t-d\widetilde{K}_t=dA_t-dK_t$. Then there must be $d\widetilde{A}_t=dA_t$ and $d\widetilde{K}_t=dK_t$, which implies $\widetilde{A}_t=A_t$ and $\widetilde{K}_t=K_t$. $\square$

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